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ON A VALUATION FIELD INVENTED BY A. ROBINSON AND CERTAIN STRUCTURES CONNECTED WITH IT

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ABSTRACT

We clarify the structure of the non-archimedean valuation field ${}^\rho\mathbf{R}$ which was introduced by A. Robinson, and of the ρ-non-archimedean hulls of Banach algebras and Lie groups. (For Banach spaces this construction is due to W. A. J. Luxemburg.) In particular, we show that any two infinite-dimensional real normed spaces have a pair of isometrically isomorphic ρ-non-archimedean hulls.

Introduction

About a decade and a half ago, Abraham Robinson defined in [25] a non-archimedean valuation field ${}^\rho\mathbf{R}$ as a residue class field of a certain convex subring of the non-standard real number system ${}^*\mathbf{R}$ (here ρ is a fixed positive infinitesimal in ${}^*\mathbf{R}$). The field ${}^\rho\mathbf{R}$ serves as a bridge between non-standard and non-archimedean analyses. Later Robinson, in cooperation with A. H. Lightstone, devoted a book [16] entirely to this field, in connection with its applications to asymptotic series. The basics of functional analysis over the field ${}^\rho\mathbf{R}$ were outlined by W. A. J. Luxemburg in [17]; here the notion of the ρ-non-archimedean hull of a normed space was introduced.

We sketch the structure of the field ${}^\rho\mathbf{R}$ in Section 1. For any internal real normed algebra A, we define its ρ-non-archimedean hull ${}^\rho A$ and study its properties (Section 2); it turns out that any two standard normed spaces of infinite dimension have a pair of isometrically isomorphic ρ-non-archimedean hulls. In Section 3, the notion of the ρ-non-archimedean hull is extended also to Banach–Lie groups.

Dedicated to the memory of A. Robinson on the occasion of the 70th anniversary of his birth.

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Throughout this paper, we follow the "classical" approach to the non-standard analysis [5,8,18].

1. The field ${}^\rho\mathbf{R}$ and its structure

(1.1) Fix a positive infinitesimal $\rho \in {}^*\mathbf{R}$ and denote by ${}^\rho\mathrm{fin}\, R$ the convex subring of all ρ-finite non-standard reals, i.e. of such $r \in {}^*\mathbf{R}$ that for some $n \in \mathbf{N}$, $|r| < \rho^{-n}$. Let ${}^\rho\mu_{\mathbf{R}}(0)$ denote the maximal (convex) ideal in ${}^\rho\mathrm{fin}\,\mathbf{R}$ consisting of all ρ-infinitesimal real numbers, i.e. of such $r \in {}^*\mathbf{R}$ that for every $n \in \mathbf{N}$, $|r| < \rho^n$. The convexity of both ${}^\rho\mathrm{fin}\,\mathbf{R}$ and ${}^\rho\mu_{\mathbf{R}}(0)$ implies that the quotient field ${}^\rho\mathbf{R} \cong {}^\rho\mathrm{fin}\,\mathbf{R}/{}^\rho\mu_{\mathbf{R}}(0)$ becomes an ordered field in a natural way. Denoting the quotient ring homomorphism ${}^\rho\mathrm{fin}\,\mathbf{R} \to {}^\rho\mathbf{R}$ by ${}^\rho\mathrm{st}$, one may totally order the field ${}^\rho\mathbf{R}$ by the rule $x \le y \Rightarrow {}^\rho\mathrm{st}\, x \le {}^\rho\mathrm{st}\, y$. The field ${}^\rho\mathbf{R}$ is real closed [16]. We denote by $\bar{\rho}$ the element ${}^\rho\mathrm{st}(\rho)$.

A non-archimedean (additive) valuation [20] on ${}^\rho\mathbf{R}$ is defined by the following formula: $v({}^\rho\mathrm{st}\, x) = \mathrm{st}\,\mathrm{Log}_\rho|x|$, $x \in {}^\rho\mathrm{fin}\,\mathbf{R}\backslash{}^\rho\mu_{\mathbf{R}}(0)$; to $v(0)$ the symbol $+\infty$ is prescribed as usual. The value group of the valuation v is precisely $\mathbf{R}^+$ [16,17].

We will use also the non-archimedean norm (= multiplicative valuation [20]) $|x|_\rho := e^{-v(x)}$ (it is assumed that $e^{-\infty} = 0$) introduced in [17].

(1.2) Remark. The valuation v agrees with the ordering of ${}^\rho\mathbf{R}$ in the sense of Gemignani [9], i.e. $0 < x < y$ implies that $v(x) \ge v(y)$.

(1.3) The residue class field of ${}^\rho\mathbf{R}$ (w.r.t. the valuation v) is the quotient field of the valuation ring $O_{{}^\rho\mathbf{R}} := \{x \in {}^\rho\mathbf{R} : v(x) \ge 0\}$ by the valuation ideal $I_{{}^\rho\mathbf{R}} := \{x \in {}^\rho\mathbf{R} : v(x) > 0\}$. We denote this field by ${}^{(\rho}\mathbf{R}$. Since both $O_{{}^\rho\mathbf{R}}$ and $I_{{}^\rho\mathbf{R}}$ are convex in ${}^\rho\mathbf{R}$ then ${}^{(\rho}\mathbf{R}$ becomes an ordered field in a way similar to ${}^\rho\mathbf{R}$. The symbol ${}^{(\rho}\mathrm{st}$ will be used to denote the residue class map $O_{{}^\rho\mathbf{R}} \to {}^{(\rho}\mathbf{R}$.

(1.4) Let K be a field and G be an ordered Abelian group. The totality of all formal power series of the form $(*)\ \sum k_\tau t^{\alpha_\tau}$, where $\{\alpha_\tau\}$ is a well-ordered subset of G and coefficients k_τ belong to K, forms a field under naturally defined operations. This field is denoted by $K\{G\}$ and called a Hahn field. (Hahn fields were introduced by Hahn [10] and have since been studied and used extensively [1,21]. For a contemporary presentation, consult [7,13].)

Every Hahn field $K\{G\}$ carries a natural valuation v defined as follows: $v(k_0 t^{\alpha_0} + \cdots) = \alpha_0$ if $k_0 \ne 0$, and $v(0) = +\infty$. Its value group is G and the residue class field is K.

If K is an ordered field then $K\{G\}$ is endowed with such an order that any element of the form t^{α}, where $\alpha \in G$, $\alpha > 0$, is a positive infinitesimal with respect to K_{+}.

(1.5) The field $K\{\mathbf{R}^{+}\}$ is denoted by $K\langle\langle t\rangle\rangle$. Let us recall some of its subfields:

(a) The field of all formal power series of the type (*) such that α_{τ}'s form an unbounded sequence is denoted by $K\langle t\rangle$. It is complete in the natural additive uniform structure [3,11,14,22] and it is real closed if and only if K is real closed [3,22]. This field inherits from $K\langle\langle t\rangle\rangle$ the valuation, with the same valuation group and residue class field.

The field $\mathbf{R}\langle t\rangle$ is denoted by $\mathbf{L}$ (in honor of T. Levi-Civita, who introduced it [15], and D. Laugwitz, who studied it extensively [14]).

(b) The field of all formal Laurent series (which means that in (*) all exponents α_{τ} are integers) with coefficients in K. It is denoted by $K((t))$. It is never real closed, but it is complete. The value group of the induced valuation is isomorphic to $\mathbf{Z}$. One may easily verify that $\mathbf{L}$ is the smallest ordered field containing $\mathbf{R}((t))$ which is both complete and real closed (= Dedekind complete, see [3,22]).

(1.6) The field ${}^{\rho}\mathbf{R}$ is spherically complete [17] (so are Hahn fields [1]). This means that any decreasing sequence of closed v-balls has a non-empty intersection.

(1.7) Remark. W. A. J. Luxemburg has mentioned (without proof) in [17], p. 196, that $\mathbf{L}$ is a spherically complete field. But this is not true. Indeed, let $B(x,\epsilon) = \{y \in \mathbf{L} : v(x-y) \geq \epsilon\}$ denote a closed v-ball. Then the following decreasing sequence of closed balls has the empty intersection:

$$B_n := B\left(\sum_{k=1}^{n} t^{-1/k}; \frac{1}{n}\right), \qquad n = 1,2,\ldots .$$

In fact, the Hahn field $\mathbf{R}\langle\langle t\rangle\rangle$ is in a sense the smallest spherically complete extension of $\mathbf{L}$.

The following result throws more light on the structure of ${}^{\rho}\mathbf{R}$.

(1.8) Theorem. *The field* ${}^{\rho}\mathbf{R}$ *is isomorphic both as an ordered field and as a valuation field to the Hahn field* $({}^{\rho}\mathbf{R}\langle\langle\bar{\rho}\rangle\rangle$.

(1.9) Remarks on the Proof. This theorem, in fact, was proved by B. Diarra in [6] (corollaire de la proposition 8), where he remarks that "c'est classique". Though Diarra's theorem was stated in a slightly less general framework (he considered, speaking our language, only those non-standard enlargements ${}^{*}\mathfrak{M}$ which can be obtained by means of ultrapowers), and the theorem above is formally a new result, but his proof can be used verbatim.

(1.10) The above theorem of Diarra strengthens a result due to A. Robinson on the embeddability of $\mathbf{L}$ into ${}^\rho\mathbf{R}$ [25]. One expects that it is possible to develop a calculus over the Hahn field $\mathbf{R}\langle\langle t\rangle\rangle$ in the same manner as has been done by Robinson for the field $\mathbf{L}$ [25].

The rest of this section is devoted to an examination of the field ${}^\rho\mathbf{R}$ as an ordered field.

(1.11) Of course, the field ${}^\rho\mathbf{R}$ is not $\aleph_1$-saturated (because it has a countable cofinality type), but the residue class field ${}^{(\rho}\mathbf{R}$ certainly is (even under the weakest of saturation-type conditions typically imposed upon the non-standard enlargement, that of sequential comprehensivity [16]).

(1.12) The field ${}^\rho\mathbf{R}$ is not exponentially closed in the sense of [2], i.e. it does not admit an order isomorphism f of its additive group with the multiplicative group of positive integers such that $f(1)$ is finite but not infinitesimal. If such a function f existed, then the element $f(\bar{\rho}^{-1})$ obviously would be larger than any element of the form $\bar{\rho}^{-n}$, $n \in \mathbf{N}$. Similarly, the residue class field ${}^{(\rho}\mathbf{R}$ is not exponentially closed. Certain difficulties arizing in the "ρ-non-archimedean Lie theory" (Section 3) originate in the non-exponential closedness of ${}^\rho\mathbf{R}$.

(1.13) A subset N of an ordered field F is called an integer set (H. J. Keisler, unpublished; see [26]) if $0 \in N \in F_+$ and for any $x \in F_+$ there is a unique $n \in N$ with $x \in [n, n+1)$. Every ordered field has an integer set [26].

Note that the field ${}^\rho\mathbf{R}$ has a natural integer set which we denote by ${}^\rho\mathbf{N}$; it is the image under the map ${}^\rho\mathrm{st}$ of the intersection of ${}^*\mathbf{N}$ with ${}^\rho\mathrm{fin}\,\mathbf{R}$. Note that ${}^\rho\mathbf{N}$ is both an additive and multiplicative subsemigroup of ${}^\rho\mathbf{R}$.

One can add to ${}^\rho\mathbf{N}$ also negative integers and obtain in such a way the subring ${}^\rho\mathbf{Z}$ of ${}^\rho\mathbf{R}$.

The ring ${}^\rho\mathbf{Z}$ is an integral domain and its quotient field ${}^\rho\mathbf{Q}$ is dense in ${}^\rho\mathbf{R}$ (this follows from the corresponding general fact, see [26]).

Warning: ${}^\rho\mathbf{Q} \neq {}^\rho\mathrm{st}({}^*\mathbf{Q} \cap {}^\rho\mathrm{fin}\,\mathbf{R})$, because the latter field coincides with the whole of ${}^\rho\mathbf{R}$ and the former does not.

2. ρ-non-archimedean hulls of normed spaces and algebras

(2.1) Let E be an internal real normed space. Following W. A. J. Luxemburg [17] (but slightly changing his notation), denote by ${}^\rho\mathrm{fin}\,E$ the set of all norm ρ-

finite elements of E. Obviously, ${}^{\rho}\text{fin}\, E$ forms a module over the ring ${}^{\rho}\text{fin}\,\mathbf{R}$; we call ${}^{\rho}\text{fin}\, E$ the principal ρ-galaxy of E. Let ${}^{\rho}\mu_E(0)$ or simply ${}^{\rho}\mu_E$ denote the set of all norm ρ-infinitesimal elements of E. Clearly, it forms a (${}^{\rho}\text{fin}\,\mathbf{R}$)-submodule of ${}^{\rho}\text{fin}\, E$; we call it the ρ-halo of zero in E. The quotient module ${}^{\rho}E \cong {}^{\rho}\text{fin}\, \mathrm{E}/{}^{\rho}\mu_E(0)$ over the ring ${}^{\rho}\text{fin}\,\mathbf{R}$ has the non-trivial annihilator

$$\mathrm{Ann}({}^{\rho}E) := \{a \in {}^{\rho}\text{fin}\,\mathbf{R} : \forall x \in {}^{\rho}E\ ax = 0\},$$

which coincides with ${}^{\rho}\mu_{\mathbf{R}}(0)$, so ${}^{\rho}E$ becomes a linear space over the field ${}^{\rho}\mathbf{R}$. The quotient homomorphism ${}^{\rho}\text{fin}\,\mathrm{E} \to {}^{\rho}E$ is denoted by ${}^{\rho}\text{st}_E$ or simply by ${}^{\rho}\text{st}$.

The linear space ${}^{\rho}E$ is endowed with a norm by the rule:

$$\|{}^{\rho}\text{st}_E x\|_{\rho} := \left|{}^{\rho}\text{st}_{\mathbf{R}}\|x\|\right|_{\rho} \qquad (x \in {}^{\rho}\text{fin}\, E).$$

This norm satisfies the strong triangle inequality and hence it is non-archimedean. It is shown in [17] that the space ${}^{\rho}E$ is spherically complete. The space ${}^{\rho}E$ is called the ρ-non-archimedean hull of E.

(2.2) Now let A be an internal normed algebra over ${}^{*}\mathbf{R}$, not necessarily associative. We assume the following condition on the norm:

$$(**) \qquad \|xy\| \le M\|x\| \cdot \|y\|, \qquad \text{where } M \text{ is a } \rho\text{-finite real number.}$$

Under the assumption (**), the principal ρ-halo of A becomes an ideal in ${}^{\rho}\text{fin}\, A$. The quotient ${}^{\rho}A$ is a ${}^{\rho}\mathbf{R}$-algebra and its norm $\|\cdot\|_{\rho}$ satisfies the property $\|xy\|_{\rho} \le |{}^{\rho}\text{st}\, M|_{\rho} \cdot \|x\|_{\rho} \cdot \|y\|_{\rho}$.

Remark that $|{}^{\rho}\text{st}\, M|_{\rho}$ is a positive real number, so the multiplication in ${}^{\rho}A$ is norm continuous. We call ${}^{\rho}A$ the *ρ-non-archimedean hull of A*.

(2.3) PROPOSITION. *Every (standard) identity which holds in A, holds also in ${}^{\rho}A$.*

PROOF. ${}^{\rho}A$ is a quotient algebra of a subalgebra of A.

For example, if A is associative then ${}^{\rho}A$ is associative; if A is a Lie algebra then ${}^{\rho}A$ also is, etc.

(2.4) Let A be a real algebra. Any of the fields $\mathbf{R}((t))$, $\mathbf{L} = R\langle t\rangle$, $\mathbf{R}\langle\langle t\rangle\rangle$, discussed in 1.5, has its analogue among extensions of the algebra A.

(a) The most well-known and (possibly) important among them is the so-called algebra of formal currents, which we denote by $A((t))$ (see, e.g., [27], ch. 4). It consists of all formal Laurent series $\sum_{n=-k}^{\infty} a_n t^n$ with coefficients a_n from A. It is an algebra over the field $\mathbf{R}((t))$.

(b) The next well-known algebra is the algebra $A\langle t\rangle$ of all Levi-Civita formal power series of the form $\sum_{n=0}^{\infty} a_n t^{\alpha_n}$, where $\{\alpha_n\}$ is an arbitrary strictly increasing unbounded sequence and $a_n \in A$. Clearly, $A\langle t\rangle$ is an algebra over the Levi-Civita field $\mathbf{L}$.

(c) Finally, the *Hahn algebra* $A\langle\langle t\rangle\rangle$ consists of all Hahn series $\sum a_\tau t^{\alpha_\tau}$ where exponents α_τ are real and form a well-ordered subset of $\mathbf{R}$ and coefficients a_τ belong to A. The Hahn field $\mathbf{R}\langle\langle t\rangle\rangle$ serves as the field of scalars.

Obviously, $A((t)) \hookrightarrow A\langle t\rangle \hookrightarrow A\langle\langle t\rangle\rangle$, and both inclusions are proper for any non-trivial algebra A.

All these algebras carry natural non-archimedean norms, defined by letting $\| a_0 t^{\alpha_0} + \cdots \| := e^{-\alpha_0}$, and they are complete w.r.t. those norms.

(2.5) Consider the algebra ${}^{(\rho}A$ over the field ${}^{(\rho}\mathbf{R}$ defined as the quotient of ${}^{(\rho}\mathrm{fin}\, A := \{x \in A : \|x\| \in {}^{(\rho}\mathrm{fin}\,\mathbf{R}\}$ by its ideal ${}^{(\rho}\mu_A(0) := \{x \in A : |x| \in {}^{(\rho}\mu_{\mathbf{R}}(0)\}$ and denote the quotient homomorphism by ${}^{(\rho}\pi_A$ or simply by ${}^{(\rho}\pi$.

(2.6) Conjecture. For any internal normed algebra A the non-archimedean normed algebras ${}^{\rho}A$ and ${}^{(\rho}A\langle\langle t\rangle\rangle$ are isometrically isomorphic. More precisely, there exists an isomorphism of valuation fields ${}^{\rho}\mathbf{R}$ and ${}^{(\rho}\mathbf{R}\langle\langle t\rangle\rangle$ such that $\bar{\rho} \leftrightarrow t$ and the algebras ${}^{\rho}A$ and ${}^{(\rho}A\langle\langle t\rangle\rangle$ are isomorphic as normed algebras over that field.

This conjecture is known to the author to be true only in the special cases represented below. Recall that any linear space can be viewed as an algebra with zero multiplication.

(2.7) Theorem. *For any internal normed linear space E, the non-archimedean normed spaces ${}^{\rho}E$ and ${}^{(\rho}E\langle\langle t\rangle\rangle$ are isometrically isomorphic.*

Proof. Pick an arbitrary isomorphism ${}^{\rho}\mathbf{R} \cong {}^{(\rho}\mathbf{R}\langle\langle t\rangle\rangle$. The desired isomorphism $i : {}^{(\rho}E\langle\langle t\rangle\rangle \leftrightarrow {}^{\rho}E$ is constructed recursively. Fix a basis of ${}^{(\rho}E$, say $\{e_\alpha\}$. Now choose $i_0(e_\alpha) \in {}^{(\rho}\pi^{-1}(e_\alpha)$ arbitrarily and extend i_0 by linearity to the linear space $E_0 := {}^{(\rho}E \otimes {}^{(\rho}\mathbf{R}\langle\langle t\rangle\rangle$ which obviously sits in ${}^{(\rho}E\langle\langle t\rangle\rangle$; it is not hard to verify that i_0 is an isometric embedding of E_0 into ${}^{\rho}E$.

Number by ordinals all elements of ${}^{(\rho}E\langle\langle t\rangle\rangle = \{a_\tau : \tau < \theta\}$. Let embeddings $i_\alpha : E_\alpha \to {}^{\rho}E$ be constructed for all $\alpha < \beta \le \theta$. If β is a limit ordinal then put $E_\beta = \bigcup_{\alpha<\beta} E_\alpha$ and define i_β in an obvious way. If $\beta = \gamma + 1$ then one can assume that $a_\gamma \notin E_\gamma$. Denote

$$A = \{\epsilon \in \mathbf{R}_+ : B(a_\gamma, \epsilon) \cap E_\gamma \neq \varnothing\};$$

here $B(a,\epsilon) := \{x \in E_\gamma : \|a - x\|_\rho \le \epsilon\}$.

Let $\epsilon \in A$; denote $\alpha = -\log \epsilon$. There exists an element $b \in E_\gamma$ such that $\operatorname{supp} a \cap (-\infty, \alpha) = \operatorname{supp} b \cap (-\infty, \alpha)$ but the coefficients in a_γ and b at t^α differ; let us denote them, respectively, by a' and b' (here $\operatorname{supp} b$ denotes the totality of all β such that the coefficient at t^β in b does not vanish). Clearly, $b + c$ belongs to E_γ; at the same time,

$$\operatorname{supp} a_\gamma \cap (-\infty, \alpha] = \operatorname{supp}(b + c) \cap (-\infty, \alpha],$$

hence there exists a real number $\alpha' > \alpha$ such that $e^{-\alpha'} \in A$. Thus, A contains a strictly decreasing coinitial sequence $\epsilon_n \downarrow A$. For any n, pick an arbitrary element $y_n \in B(a_\gamma, \epsilon_n)$ and note that

$$i_\gamma^{-1}[B(i(y_n), \epsilon_n)] = B(y_n, \epsilon_n) = B(a_\gamma, \epsilon_n).$$

Because of the spherical completeness of ${}^\rho E$, the intersection of closed balls $B(i_\gamma(y_n), \epsilon_n)$, $n \in \mathbf{N}$, contains at least one element which we choose to stand for $i_\beta(a_\gamma)$. By $E_{\gamma+1} = E_\beta$ we denote the linear span of $E_\gamma \cup \{a_\gamma\}$, and by $i_{\gamma+1} = i_\beta$ the map extended by linearity from $E_\gamma \cup \{a_\gamma\}$.

We now show that for any $z \in E_\beta$, $\| i_\beta(z) \|_\rho = \| z \|$. It suffices to verify this property for any $z \in E_\beta$ of the form $x - a_\gamma$, where $x \in E_\gamma$ is arbitrary. Now note that for any such x there is a natural number n with $x \notin B(a_\gamma, \epsilon_n)$, i.e. $\epsilon := \| x - a \| \in A$. Let n be such a number that $\epsilon_n < \epsilon$. Evidently, $\| x - y_n \| = \epsilon$, i.e. $| i_\gamma(x) - i_\gamma(y) |_\rho = \epsilon$. On the other hand, $\| a_\gamma - y_n \| \le \epsilon_n$, i.e. $\| i_{\gamma+1}(a_\gamma) - i_\gamma(y_n) \|_\rho \le \epsilon_n$. One concludes that $\| i_\gamma(x) - i_{\gamma+1}(a_\gamma) \|_\rho = \epsilon$.

This means that $i := i_\theta$ is an isometric embedding of ${}^{(\rho}E\langle\langle t \rangle\rangle$ into ${}^\rho E$. Now one easily completes the proof by constructing for every $x \in {}^\rho E$ its preimage $\sum a_\tau t^{\alpha_\tau}$ under the mapping i recursively, putting for every ordinal number τ:

$$\alpha_\tau = -\log \left\| i\left(\sum_{\sigma<\tau} a_\sigma t^{\alpha_\sigma} \right) - x \right\|_\rho \quad \text{and} \quad a_\tau = {}^{(\rho}\mathrm{st}[t^{-\alpha_\tau} x].$$

(2.8) Historically, W. A. J. Luxemburg [17] invented the notion of the ρ-non-archimedean hull of a normed space by analogy with another notion invented (by himself) earlier [18] – that of the non-standard hull of a normed space. This latter notion has become much more popular than the former one (see, e.g., the extensive survey [12]). The following problem has become a source of inspiration within this part of non-standard functional analysis: under which conditions do two given infinite-dimensional Banach spaces have a pair of isomorphic (or isometric) non-standard hulls? This problem is discussed in [12]; see also references therein.

Our aim is to show that a ρ-non-archimedean version of the above problem has a simple solution.

(2.9) COROLLARY. *ρ-non-archimedean hulls ${}^{\rho}E$ and ${}^{\rho}F$ of internal normed spaces E and F are isometric if and only if ${}^{(\rho}E$ and ${}^{(\rho}F$ are of equal dimension over ${}^{(\rho}\mathbf{R}$.*

(2.10) Before formulating the solution to the "ρ-non-archimedean hull problem", let us remark that if A is a standard Banach algebra then its ρ-non-archimedean hull does not depend on the particular choice of a standard norm on A. Indeed, let $\psi : A \to B$ be an internal homomorphism of internal normed algebras with a ρ-finite norm $\|\psi\|$. Then the formula ${}^{\rho}\psi({}^{\rho}\mathrm{st}\, x) := {}^{\rho}\mathrm{st}\, \psi(x)$ correctly defines a homomorphism ${}^{\rho}\psi : {}^{\rho}A \to {}^{\rho}B$ of the ρ-non-archimedean hulls. It is clear that ${}^{\rho}\psi$ is a bounded homomorphism with $\|{}^{\rho}\psi\| \leq |{}^{\rho}\mathrm{st}\,\|\psi\||_{\rho}$. In particular, if $\psi : A \to B$ is a standard bounded homomorphism then $\|{}^{\rho}\psi\| \leq 1$, and if ψ is a standard isomorphism of normed algebras then ${}^{\rho}\psi$ is an isometry.

(2.11) THEOREM. *Let E and F be infinite-dimensional real Banach spaces. Then there exists a non-standard model of analysis ${}^{*}\mathfrak{M}$ such that, within it, the ρ-non-archimedean hulls ${}^{\rho}E$ and ${}^{\rho}F$ are isometrically isomorphic for any positive infinitesimal $\rho \in {}^{*}\mathbf{R}$.*

PROOF. Recall an important property of regular ultrapowers: if an enlargement ${}^{*}\mathfrak{M}$ is extracted (by means of the usual procedure [5]) from a regular ultrapower $\mathfrak{M}^{I}_{U}$ of the set-theoretic structure $\mathfrak{M}$, then each internal set which is not $*$-finite has cardinality $2^{\mathrm{card}(I)}$, where I is the index set (see, e.g., [8], sec. 6, fact (3), iii).

Using the Hahn–Banach theorem, one can choose inductively a sequence $A = \{a_n\}_{n \in \mathbf{N}}$ of elements of an infinite-dimensional space E with the properties: (1) all a_n's are linearly independent; (2) $\|a_n - a_m\| \geq \delta_{n,m}$; (3) A is a bounded set.

The set ${}^{*}A \subset {}^{*}E$ is not $*$-finite, so it has external cardinality $2^{\mathrm{card}(I)}$. The properties (2) and (3) imply that ${}^{(\rho}\mathrm{st}\,|_{{}^{*}A} : {}^{*}A \to {}^{(\rho}E$ is an injection. Because of the linear independence of elements of ${}^{(\rho}\mathrm{st}({}^{*}A)$ we have: $\dim {}^{(\rho}E = 2^{\mathrm{card}(I)}$. Now it remains to apply 2.9.

(2.12) THEOREM. *The conjecture* (2.6) *is valid if A is a finite-dimensional standard algebra.*

PROOF. In this case ${}^{(\rho}A\langle\langle t\rangle\rangle$ is isomorphic to the tensor product

$${}^{(\rho}A \otimes_{{}^{(\rho}\mathbf{R}} {}^{(\rho}\mathbf{R}\langle\langle t\rangle\rangle \cong (A \otimes_{\mathbf{R}} {}^{(\rho}\mathbf{R}) \otimes_{{}^{(\rho}\mathbf{R}} {}^{\rho}\mathbf{R} \cong A \otimes_{\mathbf{R}} {}^{\rho}\mathbf{R} \cong {}^{\rho}A.$$

(2.13) THEOREM. *For any standard normed algebra A, the algebra $A\langle t\rangle$ of formal Levi-Civita series embeds into ${}^\rho A$ as a normed algebra over the Levi-Civita field* $\mathbf{L} = \mathbf{R}\langle t\rangle \hookrightarrow {}^\rho\mathbf{R}$ *(where t goes to $\bar{\rho}$).*

PROOF. Both algebras $A\langle t\rangle$ and ${}^\rho A$ contain in a clear manner normed subalgebras over $\mathbf{L}$ isomorphic to $A \otimes_{\mathbf{R}} \mathbf{L}$; besides, $A\langle t\rangle$ coincides with the completion of $A \otimes_{\mathbf{R}} \mathbf{L}$. Hence $A\langle t\rangle$ is isomorphic to the closure of a subalgebra of the complete algebra ${}^\rho A$.

(2.14) One approach to the "structurization" of infinite-dimensional Lie groups and algebras arising in contemporary mathematical physics is to make them into Frechet–Lie (Schwartz–Lie, etc.) groups and algebras (see, e.g., [19]). An alternative approach is outlined by H. N. van Eck [28]: to structurize such groups and algebras by means of embedding them into finite-dimensional Lie groups and algebras over non-archimedean valuation fields. We hope that the ρ-non-archimedean methods may contribute to the van Eck approach.

(2.15) COROLLARY. *For any standard real normed algebra A the algebra $A((t))$ embeds into ${}^\rho A$ as a normed subalgebra over the field* $\mathbf{R}((t))$, *identified with a subfield of* ${}^\rho\mathbf{R}$, $t \mapsto \bar{\rho}$.

3. Banach–Lie groups over ${}^\rho\mathbf{R}$

(3.1) Let $\mathfrak{g}$ be an internal normed Lie algebra. Since ${}^\rho\mathbf{R}$ is non-archimedean and the residue class field ${}^{(\rho}\mathbf{R}$ has characteristic zero, then the set $I_{{}^\rho\mathfrak{g}} := \{x \in {}^\rho\mathfrak{g} : \|x\|_\rho \leq 1\}$ becomes a Banach–Lie group if equipped with the binary group operation defined by the Hausdorff series. The Lie algebra $\mathrm{Lie}(I_{{}^\rho\mathfrak{g}})$ of the group $I_{{}^\rho\mathfrak{g}}$ is canonically isomorphic to ${}^\rho\mathfrak{g}$; the exponential map is defined locally around zero in the algebra ${}^\rho\mathfrak{g}$, namely, the role of it is played by the identity map. The group $I_{{}^\rho\mathfrak{g}}$ is strongly zero-dimensional and, moreover, has a neighbourhood base at its unity consisting of open subgroups. All these results follow from the general theory of Lie groups over non-archimedean valuation fields ([4], ch. 3, §4).

(3.2) We now describe how, to some extent, one may generalize the notion of the ρ-non-archimedean hull to Banach–Lie groups. We proceed in close analogy with the construction of the non-standard hull of an internal Banach–Lie group, due to us [23,24].

Let G be an internal Banach–Lie group. Fix an internal norm $\|\cdot\|$ on its Lie algebra $\mathfrak{g} \cong \mathrm{Lie}(G)$ such that $\|[x,y]\| \leq \|x\| \cdot \|y\|$ for each $x,y \in \mathfrak{g}$.

Denote by ${}^\rho\mathrm{fin}\, G$ the smallest among all, both external and internal, subgroups of G containing the set $\exp({}^\rho\mathrm{fin}\,\mathfrak{g})$. Denote by ${}^\rho\mu_G(e)$ the set $\exp({}^\rho\mu_\mathfrak{g}(0))$.

(3.3) LEMMA. *${}^\rho\mu_G(e)$ is a subgroup of the group ${}^\rho\mathrm{fin}\, G$.*

PROOF. Remember that the Hausdorff series $H(x,y)$ converges for arbitrary $x,y \in U_\mathfrak{g}$, where $U_\mathfrak{g}$ is the open ball centered at the origin of the radius $(3/2)\log 2$ in any Banach-Lie algebra $\mathfrak{g}$. Moreover, the binary operation $x \cdot y := H(x,y)$ turns $U_\mathfrak{g}$ into a local Lie group. The restriction of exponential mapping $\exp_G$ to $U_\mathfrak{g}$ is a local Lie group homomorphism (see [4]). Thus, the only thing we need is the closeness of ${}^\rho\mu_\mathfrak{g}(0)$ w.r.t. the multiplication. From the inequality [4, ch. II, §7]:

$$\|x \cdot y\| \le -\log(2 - \exp(\|x\| + \|y\|))$$

it follows that if $x,y \in \mathfrak{g}$ are both ρ-infinitesimals then so is $x \cdot y$.

(3.4) Unfortunately, the subgroup ${}^\rho\mu_G(e)$ is not, in general, normal in ${}^\rho\mathrm{fin}\, G$, even for standard finite dimensional G (for example, take the group $\mathrm{Aff}\,{}^*\mathbf{R} = {}^*\mathrm{Aff}\,\mathbf{R}$) and henceforth the direct analogy with [17,23,24] fails.

(3.5) Let us term by the *ρ-non-archimedean hull of G* the quotient group ${}^\rho G := {}^\rho\mathrm{norm}\, G/{}^\rho\mu_G(e)$, where ${}^\rho\mathrm{norm}\, G$ stands for the normalizer of ${}^\rho\mu_G(e)$ in the group ${}^\rho\mathrm{fin}\, G$. The quotient homomorphism ${}^\rho\mathrm{norm}\, G \to {}^\rho G$ will be denoted by ${}^\rho\mathrm{st}_G$ or simply ${}^\rho\mathrm{st}$.

(3.6) Following [23,24], denote by $\mathrm{fin}\, G$ the subgroup of G generated by the set $\exp_G \mathrm{fin}\,\mathfrak{g}$. The following result shows that ${}^\rho\mathrm{norm}\, G$ is non-trivial.

(3.7) PROPOSITION. $\mathrm{fin}\, G \subset {}^\rho\mathrm{norm}\, G$.

PROOF. It suffices to verify that for any $g = \exp_G y$, where $y \in \mathrm{fin}\,\mathfrak{g}$, the set ${}^\rho\mu_G(e)$ is invariant under the inner automorphism $\mathrm{Int}\, g : h \mapsto ghg^{-1}$. According to [4, ch. III, §4, coroll. 3 de prop. 8] there exists $\epsilon \in {}^*\mathbf{R}_+$ with the property: for each $x \in U_\epsilon(0)$, $g(\exp_G x)g^{-1} = \exp_G\{[\exp(\mathrm{ad}\, y)](x)\}$. Now let $h \in {}^\rho\mu_G(e)$, $h = \exp_G z$, where $z \in {}^\rho\mu_\mathfrak{g}(0)$. For some natural $n \in {}^*\mathbf{N}$ there holds $z' := n^{-1}z \in U_\epsilon(0)$. Now we have:

$$ghg^{-1} = g(\exp_G nz')g^{-1} = \exp_G\{n \cdot \exp(\mathrm{ad}\, y)(z')\}.$$

Thanks to the estimate $\|\exp T\| \le \exp\|T\|$, we have:

$$\|n \cdot \exp(\mathrm{ad}\, y)(z')\| \le n(\exp\|\mathrm{ad}\, y\|)\|z'\| \le (\exp\|y\|)\|z\|,$$

the last term being ρ-infinitesimal. q.e.d.

(3.8) LEMMA. *Let $\mathfrak{g}$ be an internal normed Lie algebra. If $x,y \in {}^\rho\mathrm{fin}\,\mathfrak{g}$, $x - y \in {}^\rho\mu_{\mathfrak{g}}(0)$ and $\exp_G tx$, $\exp_G ty \in {}^\rho\mathrm{norm}\, G$ whenever t is a finite real number, then $(\exp_G x)(\exp_G y)^{-1} \in {}^\rho\mu_G(e)$.*

PROOF. For some ρ-finite $n \in {}^*\mathbf{N}$ the elements $x' := n^{-1}x$ and $y' := n^{-1}y$ belong to $U_{\mathfrak{g}}$. One can represent $(\exp_G x)(\exp_G y)^{-1}$ as a product $\prod_{k=0}^{n} z_k$, where

$$z_k := (\exp_G x')^{n-k}(\exp_G x')(\exp_G y')(\exp_G x')^{k-n}.$$

Now the lemma follows directly from the normality of ${}^\rho\mu_G(e)$ in ${}^\rho\mathrm{norm}\, G$, and 3.3.

(3.9) Lemma 3.8 gives rise to a partially defined mapping ${}^\rho\exp\, {}^\rho\mathfrak{g} : ---\!\!\rightarrow {}^\rho G$ by the rule:

$${}^\rho\exp({}^\rho\mathrm{st}_{\mathfrak{g}}\, x) := {}^\rho\mathrm{st}_G \exp_G x,$$

whenever $x \in {}^\rho\mathrm{fin}\,\mathfrak{g}$ and $\exp_G tx \in {}^\rho\mathrm{norm}\, G$ for any finite real t.

(3.10) Due to 3.7 the domain of ${}^\rho\exp$ includes the image of $\mathrm{fin}\,\mathfrak{g}$ under the homomorphism ${}^\rho\mathrm{st}_{\mathfrak{g}}$. In particular, for $\mathfrak{g}$ standard, ${}^\rho\exp$ is defined on the naturally embedded Lie algebra $\mathfrak{g}$ itself ($\mathfrak{g} \hookrightarrow \mathrm{fin}\,\mathfrak{g} \hookrightarrow {}^\rho\mathrm{fin}\,\mathfrak{g} \to {}^\rho\mathfrak{g}$) and obviously the restriction ${}^\rho\exp|_{\mathfrak{g}}$ coincides with $\exp_G$.

(3.11) LEMMA. *Let $\mathfrak{g}$ be an internal normed Lie algebra and let $x,y \in \mathfrak{g}$ be such elements that ${}^\rho\mathrm{st}\, x$ and ${}^\rho\mathrm{st}\, y$ belong to $I_{{}^\rho\mathfrak{g}}$. Then ${}^\rho\mathrm{st}_{\mathfrak{g}}(x\cdot y) = {}^\rho\mathrm{st}_{\mathfrak{g}} x \cdot {}^\rho\mathrm{st}_{\mathfrak{g}} y$.*

PROOF. Let us denote by $H_{n,m}(x,y)$ a component of the Hausdorff series $H(x,y)$ which has full polydegree (n,m). For any standard n,m : $\mathrm{st}\, H_{n,m}(x,y) = H_{n,m}(\tilde{x},\tilde{y})$ where $\tilde{x} := {}^\rho\mathrm{st}_{\mathfrak{g}}\, x$, $\tilde{y} := {}^\rho\mathrm{st}_{\mathfrak{g}}\, y$.

Fix a positive standard r such that $\|x\|_\rho < r < 1$ and $\|y\|_\rho < r < 1$. Then clearly $\|x\| < \rho^s$, $\|y\| < \rho^s$, where $s := -\log r$ is a standard positive real number. Now let us use the inequality:

$$\|H_{n,m}(x,y)\| \leq \eta_{n,m}\|x\|^n\|y\|^m,$$

where $\eta_{n,m}$ is a standard convergent number series whose elements decrease strictly [4, ch. II, §7, n° 2, lemma 1]. It implies that

$$\|H_{n,m}(x,y)\| < \rho^{s(n+m)} \qquad \text{for any } n,m \in {}^*\mathbf{N}.$$

Let $H_k(x,y) = \sum_{n+m=k} H_{n,m}(x,y)$. It is clear that $|H_k(x,y)| < \rho^{sk}$ for any $k \in {}^*\mathbf{N}$, and that ${}^\rho\pi_{\mathfrak{g}} H_k(x,y) = H_k(\tilde{x},\tilde{y})$ for any $k \in \mathbf{N}$ standard; in particular, for such a k, $\|H_k(\tilde{x},\tilde{y})\| < e^{-sk} = r^k$. Let us denote by $H_{>k}(x,y)$ the sum $\sum_{n=k+1}^{\infty} H_n(x,y)$; remark that

$$\|H_{>k}(x,y)\| \leq \sum_{n=k+1}^{\infty} \rho^{sn} = \rho^{s(k+1)}/(1-\rho^s) < \rho^{sk}.$$

Thus, for k standard: $\|{}^\rho\mathrm{st}\, H_{>k}(x,y)\|_\rho \leq e^{-sk} = r^k$. Hence, for any standard $\epsilon > 0$ there exists a standard k with $\|{}^\rho\mathrm{st}\, H_{>k}(x,y)\|_\rho < \epsilon$, i.e.

$$\left\| \sum_{n=1}^{k} H_n(x,y) - {}^\rho\mathrm{st}_{\mathfrak{g}} H(x,y) \right\|_\rho < \epsilon.$$

This means that in ${}^\rho\mathfrak{g}$:

$$H(\tilde{x},\tilde{y}) := \sum_{n=1}^{\infty} H_n(\tilde{x},\tilde{y}) = {}^\rho\mathrm{st}\, H(x,y). \qquad \text{q.e.d.}$$

(3.12) LEMMA. *For G and norm on $\mathfrak{g}$ standard, the restriction of ${}^\rho\exp$ is a group monomorphism from $I_{{}^\rho\mathfrak{g}}$ into ${}^\rho G$.*

PROOF. It is clear from 3.7 that ${}^\rho\exp$ is defined on the whole of $I_{{}^\rho\mathfrak{g}}$. Now let x,y belong to $\mathfrak{g}$ and let ${}^\rho\mathrm{st}\, x, {}^\rho\mathrm{st}\, y \in I_{{}^\rho\mathfrak{g}}$. Since $x,y \in U_{\mathfrak{g}}$ then $\exp[x\cdot(-y)] = (\exp x)(\exp y)^{-1}$. Elements in both parts lie in fin G; therefore, one may apply to them the ρ-standard part map and obtain:

$$\rho_{exp}[{}^\rho\mathrm{st}\, x\cdot(-{}^\rho\mathrm{st}\, y)] \underset{3.11}{=} {}^\rho\exp\{{}^\rho\mathrm{st}[x\cdot(-y)]\} \underset{3.9}{=} {}^\rho\mathrm{st}_G \exp[x\cdot(-y)]$$

$$= ({}^\rho\mathrm{st}_G \exp x)({}^\rho\mathrm{st}_G \exp y)^{-1} \underset{3.9}{=} ({}^\rho\exp {}^\rho\mathrm{st}\, x)({}^\rho\exp {}^\rho\mathrm{st}\, y)^{-1}.$$

There exists a standard neighbourhood U of zero in $\mathfrak{g}$ such that the restriction $\exp_G|_U$ is an injection. Now one meets no difficulties in verifying the injectiveness of the restriction ${}^\rho\exp_G|_{I_{{}^\rho\mathfrak{g}}}$. q.e.d.

(3.13) REMARK. Generally speaking, the ρ-non-archimedean hull ${}^\rho G$ of an internal Banach–Lie group G cannot be endowed with such a ρ-non-archimedean Banach–Lie group structure that its Lie algebra is isomorphic to ${}^\rho\mathfrak{g}$ and ${}^\rho\exp$ plays the role of an exponential mapping. The simplest possible example is the following. Let $\|\cdot\|$ be the norm on the Lie algebra ${}^*u(1) = {}^*\mathbf{R}$ of the Lie group ${}^*U(1)$ defined by letting $\|x\| := M\|x\|$, $x \in {}^*\mathbf{R}$, where M is a fixed ρ-infinitely large positive number. Clearly, w.r.t. this norm: (a) ${}^\rho u(1) \cong {}^\rho\mathbf{R}$; (b) ${}^\rho\mathrm{fin}\, U(1) \cong {}^*U(1) \cong {}^\rho\mu_{U(1)}(e)$; (c) ${}^\rho U(1) \cong \{e\}$, and (d) ${}^\rho\exp$ is a trivial (constant) mapping.

(3.14) THEOREM. *Let G be a standard real Banach–Lie group, and let $\|[x,y]\|_{\mathfrak{g}} \leq \|x\|_{\mathfrak{g}}\|y\|_{\mathfrak{g}}$ whenever $x,y \in \mathfrak{g}$. Then there exists a unique Banach–Lie group struc-*

ture on the group ${}^\rho G$ such that the Lie algebra of ${}^\rho G$ is canonically isomorphic to ${}^\rho\mathfrak{g}$, and ${}^\rho\exp$ is the corresponding exponential mapping.

PROOF. Since G contains, according to 3.12, a subgroup canonically isomorphic to the ρ-non-archimedean Lie group $I_{{}^\rho\mathfrak{g}}$, then it suffices to declare $I_{{}^\rho\mathfrak{g}}$ (or, rather, its image) open in ${}^\rho G$ and to verify that for an arbitrary $y \in {}^\rho G$ there exists an open set $V \subset I_{{}^\rho\mathfrak{g}}$ such that $yVy^{-1} \subset I_{{}^\rho\mathfrak{g}}$ and the restriction $\operatorname{Int} y|_V$ is analytic. Throughout the proof we identify $I_{{}^\rho\mathfrak{g}}$ with its image in ${}^\rho G$.

Let $\tilde{y} \in {}^\rho\mathrm{norm}\, G$ be such that ${}^\rho\mathrm{st}\, \tilde{y} = y$. Let

$$S := \{r \in {}^*\mathbf{R}_+ : \forall x \in \mathfrak{g}(\|x\| \leq r) \Rightarrow \tilde{y}\exp x\tilde{y}^{-1} \in \exp U_\rho\},$$

where U_ρ stands for the ρ-ball around zero in $\mathfrak{g}$ of a unit radius.

Remark that S is an internal convex set which contains due to 3.7 the set ${}^\rho\mu_{\mathbf{R}}(0)_+$ of all positive ρ-infinitesimals. Since this latter set is external, there exists a standard natural $k \in \mathbf{N}$ such that $\rho^k \in S$. Thus, if B_{ρ^k} is the ρ^k-ball centered at the origin, then $\tilde{y}B_{\rho^k}\tilde{y}^{-1} \subset B_\rho$. This implies that $yVy^{-1} \subset I_{{}^\rho\mathfrak{g}}$, where V is the ball of radius e^{-k} centered at zero in ${}^\rho\mathfrak{g}$. From now on we identify the elements of $U_\mathfrak{g}$ with the corresponding elements of G by means of the exponentiation.

One meets no difficulties in verifying that $\|\mathrm{Ad}\, y\|$ is a ρ-finite number; according to 2.10, the linear map ${}^\rho(\mathrm{Ad}\, y) : {}^\rho\mathfrak{g} \to {}^\rho\mathfrak{g}$ is correctly defined and continuous.

Using [4, ch. III, §4, coroll. 3 de prop. 8], let us choose a standard neighbourhood W of zero in the Lie algebra $\mathfrak{g}$ such that for any $\tilde{x} \in W$, $\tilde{y}(\exp\tilde{x})\tilde{y}^{-1} = \exp(\mathrm{Ad}\,\tilde{y}\cdot\tilde{x})$. Applying to both parts of that equality the map ${}^\rho\mathrm{st}$ one obtains $y({}^\rho\exp x)y^{-1} = {}^\rho\exp({}^\rho[\mathrm{Ad}\, y]x)$ for any $x \in V$. This implies the analyticity of $\operatorname{Int} y|_V$. q.e.d.

(3.15) EXAMPLE. Let the Lie algebra ${}^*u(1) = {}^*\mathbf{R}$ of the Lie group ${}^*U(1)$ be endowed with its usual standard norm. Then ${}^\rho u(1) = {}^\rho\mathbf{R}$ and ${}^\rho U(1)$ is naturally isomorphic to the group ${}^\rho\mathbf{R}/{}^\rho Z$ with the corresponding quotient mapping coinciding with ${}^\rho\exp$; in this case, ${}^\rho U(1)$ is isomorphic to the direct sum ${}^\rho\mu_{\mathbf{R}}(0) \oplus U(1)$, the latter group being discrete, and henceforth ${}^\rho U(1)$ is a ρ-non-archimedean one-parameter Lie group.

(3.16) PROPOSITION. *For a standard normed associative unital algebra A, the ρ-non-archimedean Lie group $({}^\rho A)^\times$ of invertible elements of ${}^\rho A$ embeds canonically as an open Lie subgroup into the Lie group ${}^\rho(A^\times)$.*

PROOF. Note that the map ${}^\rho\mathrm{st}_A : {}^\rho\mathrm{fin}\, A \to {}^\rho A$ is both a Lie algebra and associative algebra homomorphism. Now observe that $\exp_A$ defines an isomorphism between the group ${}^\rho\mu_A(0)$ (endowed with the Hausdorff multiplication) and the multiplicative subgroup ${}^\rho\mu_A(1)$ of $A^\times$.

At the following step, verify that $[{}^\rho\mathrm{fin}\, A] \cap [{}^\rho\mathrm{fin}\, A]^{-1} \subset {}^\rho\mathrm{norm}\, A$ and that a group monomorphism $i\colon ({}^\rho A)^\times \hookrightarrow {}^\rho(A^\times)$ arises from that inclusion mapping.

The concluding step is to note that $i \circ \exp_{({}^\rho A)^\times} = {}^\rho\exp_{A^\times}$, whenever this equality makes sense. This demonstration proceeds very much like the proof of Lemma 3.11 above.

(3.17) Remark. The inclusion $({}^\rho A)^\times \hookrightarrow {}^\rho(A^\times)$ is a strict one. Indeed, for $A = \mathbf{R}$ the group $({}^\rho\mathbf{R})^\times$ contains an element $e^{\rho^{-1}}$, which does not belong to $({}^\rho\mathbf{R})^\times$. This implies that for any standard normed associative unital algebra A the group ${}^\rho(A^\times)$ contains an element having norm $e^{\rho^{-1}}$ and thus it does not coincide with $({}^\rho A)^\times$.

Let us write ${}^\rho A^\times$ instead of ${}^\rho(A^\times)$.

(3.18) A group of formal currents $A((t))^\times$ consists of all invertible elements of the algebra $A((t))$ of formal currents with coefficients in an associative unital Banach algebra A. This is a Banach–Lie group over the field $\mathbf{R}((t))$.

(3.19) Corollary. *Let A be a standard normed associative unitary algebra. Then the group of formal currents $A((t))^\times$ embeds as a Lie subgroup into the group ${}^\rho A^\times_{\mathbf{R}((t))}$, obtained from the Lie group ${}^\rho A^\times$ by narrowing the scalar field from ${}^\rho\mathbf{R}$ to $\mathbf{R}((t))$, $t \leftrightarrow \bar{\rho}$.*

Proof. Since $\mathbf{R}((t))$ is a non-discrete valuation subfield of ${}^\rho\mathbf{R}$, then one may apply [4, ch. III, §1, example 3] to both groups ${}^\rho A^\times_{\mathbf{R}((t))}$ and $({}^\rho A)^\times_{\mathbf{R}((t))}$ and then use 3.16.

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